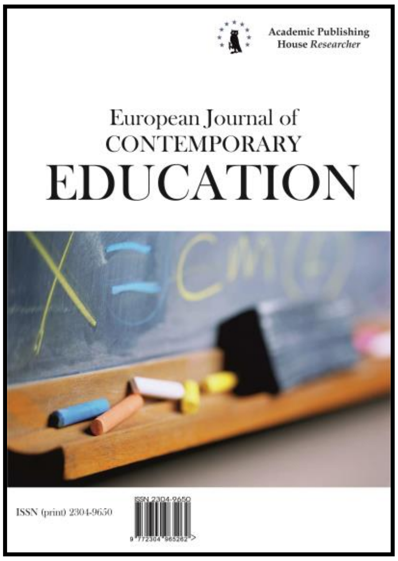
Published in the Slovak Republic
European Journal of Contemporary Education
E-ISSN 2305-6746
2019, 8(1): 52-68
DOI: 10.13187/ejced.2019.1.52
www.ejournal1.com



Visualisation as a Method for the Development of the Term Rectangle for Pupils in Primary School

Ján Gunčaga [a, *], Katarína Žilková [a]

[a] Faculty of Education, Comenius University, Bratislava, Slovak Republic

Abstract

Visualisation is an important part of the cognitive process in mathematics. Pedagogical-psychological theoretical frameworks concerning cognitive processes consider perception and visualisation as basic conditions for the creation of correct mental imagery. This study examined visualisation as a key factor in the development of geometric concepts for pupils in primary and lower secondary education. The study has empirical-applicational characteristics. The results of research on Slovakian students regarding geometric concepts about rectangles and squares are described. The research sample consisted of two groups of pupils in the 4th grade (primary level) and 9th grade (lower secondary level). The comparison of results between the two groups enabled the identification of several misconceptions made by pupils regarding rectangles and several suggestions are included for educational interventions based on static and dynamic visualisation models.

Keywords: visualisation, pupils, primary school, results.

1. Introduction

The visualisation of geometric terms and procedures is an important element in the process of creation of geometric conceptions. Basic pedagogical and psychological cognitive models, which have been enhanced to provide descriptions of cognitive processes in mathematics or geometry, indicate the relevance of visualisation processes. Based on the results of the present study on the geometrical conceptions of Slovakian pupils in the 4th grade of primary school and the 9th grade, and their comparison, we identified the most frequent misconceptions regarding planar shapes. On-going misconceptions were found in themes regarding rectangles, squares and their properties. Thus, in this study, real and virtual education models that have the potential to create desirable conditions for the creation of tangible conceptions or for the correction of misconceptions of geometric terms and their properties were established (see also Žilková et al., 2018).

* Corresponding author
E-mail addresses: jguncaga@gmail.com (J. Gunčaga)

2. Discussion

Theoretical framework: visualisation as a component of the cognitive process

The meaning of visualisation in the cognitive process is supported by many educational and psychological theories. In the following section, several examples of cognitive theories are presented. Visualisation plays an important role in these theories in the field of cognitive processes.

Swiss psychologist Jean Piaget (1896–1980) studied the cognitive development of children and examined the process of creation of their mental models. Piaget's concept of cognitive development of a child defines four stages that bond to the biological development of a child (McLeod, 2018): sensorimotor stage (birth to age 2), pre-operational stage (from age 2 to age 7), concrete operational stage (from age 7 to age 11) and formal operational stage (age 11+). Piaget and Inhelder (2010) studied the creation of figurative thoughts and compared their qualities at the pre-operational and operational levels. They distinguished between two types of figurative thoughts: reproductive imagery—imagery of known and earlier observed events, and anticipatory imagery—imagery that illustrates movements or transformation. They found that on a pre-operational level only static figurative imageries are present. Thus, children are able to create anticipatory imagery only after they reach the level of real operations (approximately from 7 years of age). Therefore, if we expect from a child an imagery of movement or some transformation, then this process requires imageries of anticipatory character.

If we want to reflect the results of Piaget and Inhelder's (2010) research to the teaching of geometry, it is necessary to create educational interventions in which conditions for the creation of static figurative imagery are made first and only then the activities can be expanded to examine the dynamic models of geometric terms and procedures. For example, the examination of static models and non-models of squares (e.g. pictures of shapes, puzzles, creation of models of squares in a square grid, geoboard or with the use of sticks, folding and cutting paper and so on) must be made before the creation of mental models regarding the geometric term 'square'. Thus, children gain experiences on which shape is and which shape is not a square. Several software programs of dynamic geometry (e.g. GeoGebra) may be appropriate environments for the creation and examination of dynamic models of squares. Within these educational software programs, it is possible to visualise the dynamic and transformation processes that are difficult to realise with static models. Thus, software programs can support the creation of anticipatory imagery in cases where a child is not able to mentally manipulate geometrical imagery.

American psychologist Jerome Seymour Bruner (1915–2016) studied cognitive processes from the point of view of intellectual development. Within his cognitive model, Bruner (1966) suggested the application of the following three types of representations: ***enactive representations, iconic representations and symbolic representations***. Enactive representations are principally based on perception, which means manipulation with concrete objects. Iconic representations assume working with imagery of mental thoughts that a child has already encountered. The highest level of abstraction is symbolic representations in which the ability to describe terms based on valid professional terminology is assumed and imageries are qualitatively on a higher level. Bruner's (1966) theory is effective when we proceed from enactive through iconic to symbolic representations in education. Bruner's (1966) theory states that the creation of a child's conception is conditioned by education and the influence of adults where it is necessary to use the correct type of representation. What then are the educational implications for geometric education in the context of Bruner's (1966) theory? It is evident that the visualisation of geometric terms with the use of enactive and iconic representations determines the creation of correct geometric conceptions. If a geometry teacher accepts the biological and cognitive development of each child individually, then he/she creates conditions for the creation of correct abstract conceptions regarding geometric terms. If some of the stated conditions are not reflected in the educational process, then misconceptions about geometric terms might be created that become obstructions to further geometrical recognition (Žilková, 2017).

The cognitive process in mathematics and its specifics were elaborated by Hejný (2014) and named the 'Theory of generic model' (TGM). TGM consists of five stages (***motivation, isolated models, generic modes, abstract cognition and crystallisation***) and two cognitive transformations (***process of generalisation and process of abstraction***). The term 'generic model' encorporates the entire set of different isolated models. Hejný's (2014) theory reflects the cognitive process stages in mathematics but also accentuates the importance of many different experiences. The specification of geometric terms is also based first on manipulation with concrete models and non-models of geometric terms creating conditions for the creation of abstract representation of terms (generic model) later on. For a child, the generic model is also a concrete representative of a certain group of isolated models. Abstract conceptions about geometric terms are no longer connected with a concrete model but integrate all important elements and properties of a given term. Thus, visualisation is more a representation of the stages of motivation, isolated models and generic models within TGM.

Teachers working in the school environment seldom use interactive approaches during the introduction of geometric notions. Haptic and virtual manipulation can help pupils understand these geometric notions. Manipulative activities are oriented towards giving pupils experience with several selected plane shapes in different positions and different magnitudes. It is important that the personal and figural concepts (Fischbein, 1993) are not connected to only one model of a particular shape (for example in the standard size).

One of the best known theories regarding the development of geometric terms is the van Hiele model of geometric thinking. Pierre van Hiele (1986, 1999) observed and defined five different levels of geometric thinking of individuals: ***visualisation, analysis, abstraction, deduction and rigour***. Each level has its own characteristic expressions based on those that can identify the level of geometric thinking of a child. The van Hiele theory has been validated in previous studies and elaborated on with the goal of providing supportive evidence, complete characteristics of levels and suggesting educational interventions (e.g. Al-ebous, 2016; Burger, Shaughnessy, 1986; Clements, Swaminathan et al., 1999; Çontay, Paksu, 2012; Crowley, 1987; Erdogan, Dur, 2014; Feza, Webb, 2005; Knight, 2006; Mason, 2002; Mayberry, 1983; Musser et al., 2001; Usiskin, 1982; Van de Walle, 2001).

In our study, the importance of visualisation during the process of development of geometric terms is evident directly from the name of the zero van Hiele level. According to the van Hiele theory, the level of visualisation is characterised by the fact that a child is able to identify a geometric shape based on its visual prototype. This level is not typical of a child's own thinking but is important within the entire process. It creates conditions for another level of analysis within which one is able to distinguish important elements and properties of geometric shapes. If the provision of experiences on both levels during the educational process is omitted then the child does not have the opportunity to develop geometric thinking in more abstract forms. Educational implications of the van Hiele theory follow the phases of the instructional cycle: ***information, guided orientation, explication, free orientation and integration*** during the teaching of geometry. Details and experiences with the phases of the instructional cycle have been described in previous studies (e.g. Al-ebous, 2016; Crowley, 1987; Dongwi, 2014; Mason, 2002; Usiskin, 1982).

Jirotková (2010) provides reasons for the importance of visual imagery and states that the core of geometry is the relationship between geometric objects. Simultaneously, the conception regarding geometric objects is created in the mind via visual and tactile imagery.

When dealing with the theme of visualisation, the theory of French psychologist Duval (1995) must be mentioned, who identified reasons "why" and "how" geometry should be learnt or taught in schools. He formulated three types of cognitive processes that must be supported within the development of geometric thinking:

A) ***Visualisation processes*** in which transparent visual illustrations or representations of geometric propositions (e.g. pictures, graphs, symbols) are present; heuristic exploration of a complex geometric situation.

B) ***Construction processes*** include the use of construction tools (e.g. ruler, bow compass or dynamic software programs) for solving a geometric situation, thus creating a geometric model based on certain conditions.

C) ***Argumentation processes*** mean specific logic discursive processes for the generalisation and development of knowledge, explanation, reasoning and proofs.

These three processes may occur separately but they are also closely linked together and synergic, that is, according to Duval (1998), cognitively indispensable for acquiring geometric knowledge. Visualising means summoning up a mental image of something—seeing it in your mind (Using visualization in maths teaching, 2018a, 2018b).

From the above stated theoretical cognitive models, it is evident that ***visualisation determines the creation of correct geometric conceptions***. Therefore, it is important to implement the correct, most effective and age appropriate choice of visual models to access geometric terms within the teaching of geometry.

Research basis: conceptions of Slovakian pupils regarding rectangles

Mathematical terminology in Slovakia is different from foreign terminology in the definitions of some concepts. In Slovakian school mathematics there are several exclusive definitions. Many pupils during their mathematics education for the entire primary level have not had any experience with the notion of a "rectangle". Some teachers actively use the notions of oblong and square and they do not use the common notion, i.e. "the rectangle" for this type of shape. Other teachers do use the notion "rectangle"; however, this concept isnot included in the Slovakian national curriculum. The oblong is defined, such as a parallelogram, which is not a square and every inner angle is a right angle. The square is defined as a regular quadrilateral. For this reason, it is difficult to adapt research tools from overseas to the Slovakian conditions in the relevant form, which makes it difficult to compare research results from Slovakia and abroad. Therefore, we must consider the specific terminological conditions of each country.

From 2015 to 2017 we investigated the level of knowledge in geometry of pupils in different age groups within the project VEGA (Scientific Grant Agency of the Ministry of Education, Science, Research and Sport of the Slovak Republic and the Slovak Academy of Sciences) and APVV (Slovak Research and Development Agency). In the present study, we focused only on the description of selected results of the research concerning geometric conceptions of pupils regarding rectangles and those that have connection with the visual concepts of pupils. The results from the present study inspired the creation of educational interventions focused on the correction of long-lasting misconceptions about rectangles. The aim of the present study was to determine what misconceptions pupils have in primary schools about geometric shapes, identify the most common misconceptions and determine whether these misconceptions are minimised during the geometric education process or whether they persist because of their fixity and stability. The research sample was split into two subsamples. One subsample consisted of approximately 345 pupils in the 4th grade from 26 primary schools and approximately 10 years of age (52.5 % boys and 47.5 % girls). This subsample was a convenience sample from north and northeast Slovakian schools. Pupils solved the test of their mathematical knowledge. Administrators were trained in the organization of this test and at each of the 26 schools the administrator arranged the correctness of the data.

The second subsample consisted of 738 Slovakian pupils from the 9th grade of primary schools or the 4th grade of 8-years secondary schools (15 years old pupils, upper secondary level). The pupils were approximately 14–15 years old and 52 % were boys. This was a representative sample. The members of the research team personally administrated the objectiveness of the sampled data at every selected school without the influence of the teachers of the tested pupils.

To determine the conceptions of pupils in the 4th grade of primary school about geometric shapes and their properties, we used our non-standardised knowledge test. The conception content of the test agrees with the actual content and performance standards of the National Educational Programme (2009) and also with the Innovated National Educational Programme for Primary Education (2015) in Mathematics and work with information. These types of research tools have been applied in previous studies (for example Burger, Shaughnessy, 1986; Clements, Sarama, 2014; Hannibal, 1999; Levenson et al., 2011 and others). Some of the tasks of our test inspiration were obtained from research by the Erikson Institute (2013). Tasks on our test respected Slovakian educational conditions.

To determine the conceptions of the 15-year old pupils we used a component of van Hiele's test created for the needs of the Cognitive Development and Achievement in Secondary School Geometry (1982) project by Zalman Usiskin[*] (see Usiskin (1982)). We added five of our own tasks to obtain more accurate results because of the differences in national terminology.

The following section describes the most important results from both studied groups concerning the conceptions of pupils about rectangles in the context of visualisation, focussing on

[*] "Copyright ©1980 by the University of Chicago. Reprinted with permission of the University of Chicago (Prof. Z. Usiskin)."

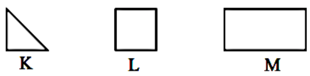
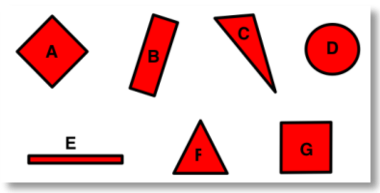
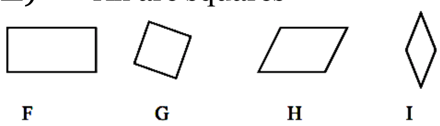
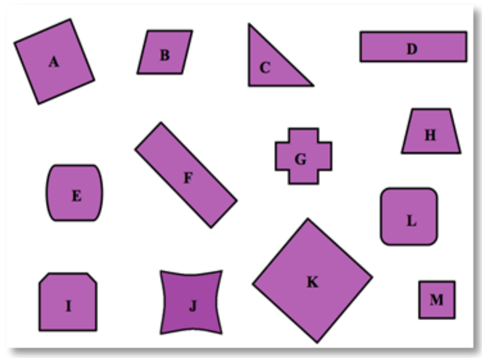
comparable and age accurate tasks used in both groups. The most frequent problems in the identification of rectangles in both groups of pupils are specified and the types of problems and misconceptions common to both groups are discussed. When determining the common problems among these groups, it was expected that the early misconceptions of pupils would be stable during their learning at both the primary and lower secondary levels. Teacher should determine educational interventions and teaching activities that eliminate the creation of these misconceptions during the early stages of the pupils' cognitive processes.

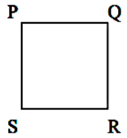
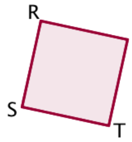
We aimed to identify the concepts of plane figures that exist for pupils in the 4th and 9th grades and whether the educational interventions of teachers in the lower secondary level eliminate the misconceptions about these plane figures. Some models of rectangles in rotated or other non-standard positions are difficult for pupils to comprehend; thus, teachers should make suitable educational interventions in these cases.

The tasks aimed at squares for the van Hiele's level of visualisation (tasks L1.1, L1.4, S1.1 and S1.2) and the analyses that contained comparable items (tasks L2.1 and S2.8) are listed in Table 1. The tasks that are denoted with the letter L are from the original test prepared by Zalman Usiskin and the tasks denoted with the letter S are from our test. The formulation of the tasks L1.1 and S 1.1 was different for the different groups of pupils. We focused on whether the pupils in both groups had the ability to identify squares. Those pupils in the 9th grade had the possibility of choosing the square because they already knew this shape; thus, we wanted to examine the ability of the pupils in the 4th grade to identify squares.

However, some tasks for these two groups of pupils were not comparable, thus we determined these via parallels in the cognitive process for pupils from both groups. For example, if the task L1.1 is oriented to the identification of squares, then we evaluate in the task S1.1 for purposes of our study only the shapes that are squares. Therefore, in task L1.1 the possibility B responds to the possibility G in task S1.1.

Table 1. Tasks concerning squares at the level of visualisation and analysis for pupils in the 4th and 9th grades

Pupils in the 9th grade	**Pupils in the 4th grade**
L1.1 Which of these are squares? A) K only B) L only C) M only D) L and M only E) All are squares 	S1.1 Write a name for each shape. 
L1.4 Which of these are squares? A) None of these are squares B) G only C) F and G only D) G and I only E) All are squares 	S1.2 Which of the shapes in the picture are squares? 

L2.1 PQRS is a square. Which relationship is true in all squares? A) PR and RS have the same length. B) QS and PR are perpendicular. C) PS and QR are perpendicular. D) PS and QS have the same length. E) Angle Q is larger than angle R. 	S2.8 STUR is a square. Circle true statements: A) Line segment RT is a side of square. B) Side SU is a diagonal of square. C) Sides RU and RT are adjacent. D) Sides SR and TU are opposite. E) Sides ST and TU have different length. F) Opposite sides in a square have same length. G) Line segments SU and SR have same length. 

The percentage share of answers for each possible answer in tasks L1.1 and L1.4 for pupils in the 9th grade are shown in Table 2. The correct answer for both tasks was B. Code 'NA' represents the situation when a pupil did not answer the task or chose multiple answers. If we compare the correct answer (G) chosen by the 4th graders for task S1.1 and the correct answer (B) chosen by the 9th graders for task L1.1, then approximately 92 % was obtained in both groups, with the percentage obtained by the 9th graders higher than that of the 4th graders.

We also compared the ability to identify a square in the rotated position—the square in task L1.4 (shape G identified by 75.5 % of 9th graders) and the square in task S1.1 (shape A identified by 54.8 % of 4th graders)—by the 4th and 9th graders.

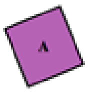
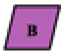
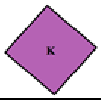
Table 2. Percentage share of answers of 9th graders for all possible answers in tasks L1.1 and L1.4 – identification of a square at the visualisation level

Task	A (%)	B (%)	C (%)	D (%)	E (%)	NA (%)
L1.1	1.22	92.28	1.08	5.01	0.00	0.41
L1.4	5.96	75.47	4.88	10.98	2.17	0.54

Both these tasks verified the level of knowledge of 9th graders in the identification of squares. Task L1.1 was relatively simple for the 9th graders, with 92.3% of pupils answering correctly and none of the pupils choosing answer E. Distractors A and C were also not very popular. The correct answer in task L1.4 was B; however, only 75.5 % of 9th graders chose this answer. A popular distractor for pupils was answer D (11 %); therefore, the pupils identified the rhombus 'I' as a square. Approximately 6 % of pupils chose distractor A; therefore, they were unable to identify a square in the rotated position.

For comparison, we chose the results of analogous tasks and comparable items from the test of pupils in the 4th grade, which are shown in Table 3. The lowest success rate of 4th graders was in the identification of models of squares in which the diagonals were in a horizontal-vertical position. The success rate for task S1.2-B was 51 %; therefore, 49 % of the 4th graders identified a rhombus as a square. The term 'perpendicularity' is not present in the national curriculum or in mathematics textbooks for primary education and pupils do not encounter this exact property (only as an intuition). Therefore, they probably thought that the rhombus in S1.2-B was a shape that was a quadrilateral and had all sides equal, with these characteristics for a 4th grader being enough for the identification of the shape as a square.

Table 3. Success rate of correct answers of pupils in the 4th grade– identification of a square at the visualisation level

Shape	S1.1-A 	S1.1-G 	S1.2-A 	S1.2-B 	S1.2-K 	S1.2-M 
Answer (%)	54.8	91.9	91.0	51.0	80.0	95.4

Active knowledge of terminology was expected from the 4th graders in task S1.1. Pupils should in this task be able to name the shapes of the figure (see task S1.1 in Table 1). If we formulate the task in this way (i.e. active terminology), then this task is much more difficult than the task where the pupils work only with passive terminology. Therefore, the results from this task were expected to be worse than the results from task S1.2.

If we compare the results between tasks S1.1-A and S1.2-K, than this is true, which is the same if we compare the results between tasks S1.1-G and S1.2-M. Pupils found it more difficult to write the name of the shape than to identify the shape according to its name.

Comparing the results between tasks S1.2-A and S1.2-K, the measure of the rotation of certain shapes influenced the pupils' ability to identify the name of the shape. A smaller angle of rotation ensured greater success for pupils when identifying these shapes.

Comparing the results from the analogous parts of tasks for pupils in the 4^{th} and 9^{th} grades at the van Hiele's level of visualisation (tasks L1.1, L1.4, S1.1 and S1.2) we observed an expected increase in correct answers for the 9^{th} graders. However, in both groups there was a lower success rate for the identification of squares that did not have their sides in the horizontal-vertical position. Thus, the degree of rotation of the square affected the ability of the pupils to identify the squares. The most difficult shape to identify for both groups were those squares that had their diagonals in the horizontal-vertical position.

The percentage share of answers of 9th graders for all possible answers to task L2.1 are listed in Table 4. The correct answer for this task was B.

Table 4. Percentage share of answers of 9th graders for task L2.1

Task	A (%)	B (%)	C (%)	D (%)	E (%)	NA (%)
L2.1	12.47	46.21	24.12	14.63	1.22	1.36

Table 4 shows that the correct answer regarding the properties of a square was chosen by only 46 % of 9th graders. This is a relatively poor result given that the task only tested simple properties and relationships in squares. A relatively popular distractor was answer C, with approximately 24 % of 9th graders denoting that the opposite sides of a square are perpendicular. We were unable to determine whether the mistake was real or not, i.e. whether the pupils confused the terms perpendicular and parallel or whether it was really a misconception.

Two distractors, answers A and D, focused on verification of the ability of pupils to compare the length of a side and a diagonal in a square. Pupils in the 9th grade considered these lengths to be the same in approximately 12 % (distractor A) and 15 % (distractor D) of cases. Therefore, more than a quarter of tested 9th graders chose either distractor A or D as the correct answer, and thus they believed that the lengths of a side and a diagonal in the square are the same.

The formulation of the task for the 4th graders was different from the formulation of the task for the 9th graders. For the 4th graders, we verified whether they knew the basic terms concerning squares (e.g. side, diagonal, adjacent and opposite sides) and we also included the properties about the length of the side and diagonal of a square. The adjustment of this task was required because Slovakian 4th graders do not know the property 'perpendicular' because it is not included in the national curriculum.

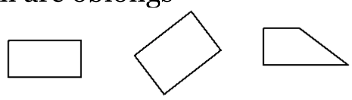
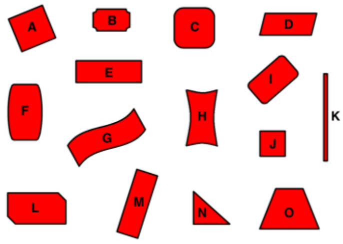
The percentage of answers of 4th graders for the statements A to G in task S2.8 (the last row in Table 1) are shown in Table 5. Approximately 80 %–90 % of 4th graders knew the basic elements of a square (e.g. side, diagonal and adjacent sides) and they knew the property about the equal length of the sides of a square. A total of 75 % of the 4th graders also knew the term 'opposite sides' (distractor D). Approximately 55 % of the 4th graders thought that the length of the side and the diagonal in a square was the same.

Table 5. Percentage of answers of 4th graders to statements in task S2.8

Statement	A	B	C	D	E	F	G
Answers (%)	82.6	83.7	82.6	75.6	82.9	87.8	54.5

Comparing the results of 4th and 9th graders showed that the most difficult item for both groups was to decide about the length of a side and diagonal in a square. The success rate for 9th graders for this item was higher than the success rate for 4th graders, which was expected, although misconceptions were identified in more than a quarter of 9th graders.

Table 6. Task about oblongs at the level of visualisation for pupils in the 4th and 9th grades

<table>
<tr><th>Pupils in the 9th grade</th><th>Pupils in the 4th</th></tr>
<tr><td>L1.3 Which of these are oblongs*?
A) S only
B) T only
C) S and T only
D) S and U only
E) All are oblongs

</td><td>S1.4 Circle which shapes <u>are not</u> oblongs.

</td></tr>
</table>

In Table 6, the tasks at the van Hiele level of visualisation focused on oblongs (tasks L1.3 and S1.4). (Remark: in the Slovakian version of the original task L1.3 of the van Hiele test the word 'rectangle' was replaced by the word 'oblong' that means rectangle, i.e. is not a square. The reason for this replacement is the different national terminology.)

The percentage share of answers for the 9th graders for task L1.3 are shown in Table 7. The correct answer was answer C. The identification of an oblong at a visualisation level was not difficult for the 9th graders, although 5.56 % thought that the oblong was the shape that had its sides in the horizontal-vertical position (distractor A).

Table 7. Percentage share of answers of 9th graders for task L1.3

Task	A (%)	B (%)	C (%)	D (%)	E (%)	NA (%)
L1.3	5.56	0.54	92.28	0.68	0.81	0.14

The oblongs in task S1.4 were the shapes E, K and M, with the success rate of pupils in the 4th grade choosing these shapes shown in Table 8.

Table 8. Success rate of 4th graders choosing shapes E, K and M for task S1.4

[2]The term rectangles in the test by prof. Usiskin has been replaced by the term oblongs because the national terminology and national curriculum in Slovakia is different than English terminology and we remarked this fact in the beginning of the part 3.

Shape	E 	K 	M 
%	94.0	87.0	92.0

Comparing the results between the 4th and 9th graders in the task that focused on the identification of an oblong based on a picture showed that the conceptions of both groups were comparable. We did not find any significant progress in 9th graders compared to 4th graders.

Because 9th graders understood the term 'rectangle', we gave them one more task where we tested which shapes were understood by them as being a rectangle.

Task: Which shape in the picture is a rectangle?

A) Only *L*.
B) Only *K* and *N*.
C) Only *L* and *M*.
D) Only *K*, *L* and *M*.
E) All of them are rectangles.

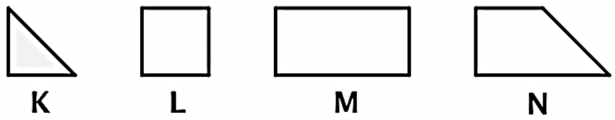


Table 9 shows the percentage share of answers of 9th graders to all possible answers of this task.

Table 9. Percentage share of answers of 9th graders for all possible answers in task about rectangles

Answer	A (%)	B (%)	C (%)	D (%)	E (%)	NA (%)
Term rectangle	3.79	14.91	40.51	11.65	26.56	2.57

The correct answer, C, was chosen by only 40.51 % of pupils. A high percentage of pupils chose distractor E as their answer; therefore, 26.56 % of 9th graders considered all shapes in the picture as being rectangles. Results showed that pupils in the 9th grade did not have an exact conception about the term 'rectangle'. They often considered planar shapes that had at least one right angle as being rectangles. The cause of this may be the fact that 9th graders have problems with understanding the quantifier in the definition of rectangle (i.e. rectangle is a planar shape that has all inner angles as right angles) or they have created a misconception of what is and what is not a rectangle at the level of visualisation.

Findings and Discussion

Following is a summary of the problems obtained for pupils in the 4th and 9th grades in understanding the terms 'rectangle' and 'square':

- **A problem with the identification of a square in the context of its position.** The position of a square had an effect on its identification, especially for 4th graders. The degree of rotation was also an important attribute. A square, whose diagonals are in the horizontal-vertical position, was the most difficult to identify for both groups. Thus, we propose educational interventions in the set of activities for pupils where they can manipulate squares and work with rotated squares.
- **A problem with the distinction between squares and rhombuses.** A model of a rhombus was denoted as a square by 46 % of 4th graders and 11 % of 9th graders. Educational interventions to assist with this problem should include specific haptic and virtual activities. Pupils should obtain experiences with squares and rhombuses, and thus determine the differences between these shapes, as well as their common properties.
- **A problem with conceptions about the length of a side and diagonal of a square.** Pupils considered that the side and diagonal of a square were equal. These conceptions were shown in 55 % of 4th graders and 27 % of 9th graders. It is important to implement into mathematics lessons more activities oriented to recognise the different properties of squares and their important elements.

• **A problem with conceptions about the term rectangle.** Only 41 % of 9th graders were able to identify a rectangle at the level of visualisation (the term rectangle could not be examined in 4th graders because it is not a part of the national curriculum). This result is connected to the use of the term "rectangle" in Slovakian school mathematics. Thus, activities should be devoted to inquiry activities for pupils, which allow them to find inclusive relationships between squares and rectangles and also the relationships between oblongs and rectangles.

Our findings correspond with the results of other research studies (for example Burger, Shaughnessy, 1986; Clements, Sarama, 2014; Clements et al., 1999; Levenson et al., 2011 and many others).

However, the studied pupils, i.e. 4th graders and 9th graders, have different educational surroundings, and we observed analogous problems in geometrical thinking by both groups. We expect that the incorrect concepts created in the younger school age pupils influence geometrical thinking of adults (also in future teachers of mathematics at the different types of schools). Žilková (2013) identified analogous problems by students in her study—future teachers at pre-primary and primary level. She argued that their misconceptions are very stable and very difficult to eliminate. Some roots of these problems are the creation of false concepts during early childhood, and an insufficient number of suitable educational interventions from teachers at school.

A study by Erdogan and Dur (2014) showed that "the preservice mathematics teachers' knowledge of quadrilaterals learnt at primary-secondary school level and prototypical images were dominant in their personal figural concepts". Their findings underline the fact that prototypical images affected their personal figural concepts. The same findings have been shown in studies by Marchis (2012) and Ҫontay and Paksu (2012).

One possible reason for the creation of a false conception by pupils is insufficient educational intervention from teachers oriented to the manipulation and working with different models of shapes in different figural, positional and metrical variations. The goal of these interventions is to support the visual perceptions of pupils, which can enable better identification of shapes as planar figures that are represented not only via their prototypes. The results of the study by Clements et al. (1999) introduced the term "pre-recognition level" in the cognitive development of pupils. This study detected the fact that pupils describe shapes based on visual forms and it was possible to see individual differences in the argumentation of different pupils.

Our research did not aim to formulate any final meanings regarding pupils' thinking about geometrical shapes, which is a limitation of our research tool. We attempted to recognise some problems with the identification of shapes that were common in 4th and 9th graders. These findings are important for the educational environment because the identification of problems in the learning process provides the potential for solving these problems. The following section proposes several traditional and non-traditional educational interventions that have the potential to improve conceptions about geometric terms and relationships in rectangles.

Educational interventions: reflection on research findings via perception activities and dynamic visualisation models

The results of the present study showed problems associated with the misconceptions of pupils regarding rectangles and squares. Early education intervention may help to change misconceptions or to improve established conceptions. Therefore, mathematics teachers must include activities into their educational processes that help eliminate existing or potential problems.

In the previous section, we analysed the content of pupils' answers in two groups containing 4th and 9th graders. We identified the most common misconceptions in these two groups of pupils. In the following section, we will focus on proposals of particular activities for mathematics teachers that can help in the elimination of the above formulated misconceptions. An important role in the presentation of these educational visualisation activities is the use of appropriate software programs (e.g. GeoGebra).

In both groups of pupils, a problem with the identification of squares in non-standard positions (not with sides in a horizontal and vertical position) was seen; thus, it is necessary to create an educational environment in which pupils may encounter models of squares in different positions.

For example, it is possible to use a real or virtual geoboard to model squares of different sizes and in different positions. The aim of these activities is for pupils to realise that the change of the square's position does not mean a qualitative change in shape to another shape (i.e. a square does not stop being a square). This is shown in the following tasks, with solutions illustrated in Figure 1a and 1b.

Task 1. Model two squares on the geoboard such that their common parts are a triangle.
Task 2. Model two squares on the geoboard such that their common parts are a square.

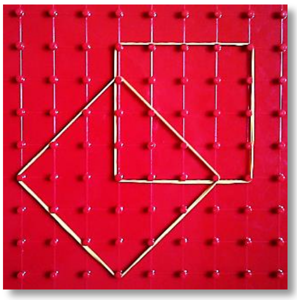


Fig. 1a. Illustration of solution to Task 1

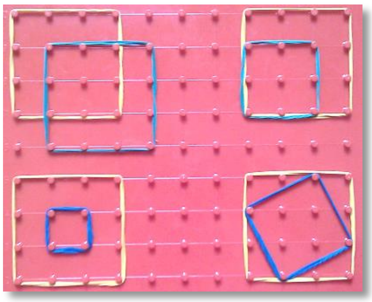


Fig. 1b. Illustration of solution to Task 2

The aim of teaching mathematics effectively is to reflect the different needs, abilities and interests of the pupils. Therefore, we recommend that a diversity of environments be used in which the activities occur. Supplementing activities from different environments ensure that the conceptions of pupils are specified. To model a square in the geoboard may be a simple activity for a particular pupil but the same task undertaken in another environment (e.g. square paper) may cause him/her problems. For another student, this might be the opposite. These observational results were collected directly during research on pupils in primary education. Therefore, an alternative task was undertaken that aimed to investigate the properties of a square after changing its position, for example, a task performed in a dynamic geometry environment or on a (virtual or printed) square grid (Figure 2).

Task 3. Draw in the square grid (using the GeoGebra software program) different squares that have vertices in grid points of a square grid.

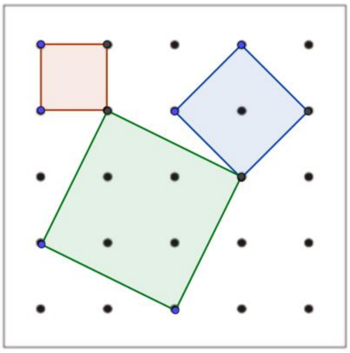


Fig. 2a. Illustration of solution to Task 3 on a virtual geoboard (Source: https://www.geogebra.org/m/wpxuvjyq)

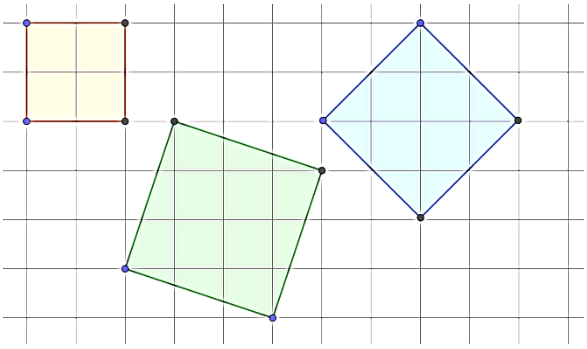


Fig. 2b. Illustration of solution to Task 3 in the GeoGebra software program environment (Source: https://www.geogebra.org/m/k5gzqebf)

To determine other properties that are involved in the process of distinguishing between squares and oblongs, it is possible to use traditional methods characteristic of mainly lower age groups of children, such as paper folding. A pupil in a lower grade is able to fold and cut a sheet of paper to create a square. In principle, this is a learnt algorithm without reasoning the sequence of instructions. In higher grades, it is necessary to discuss the reasoning behind the algorithm, thus reasoning each step involved in the folding and cutting (Figure 3).

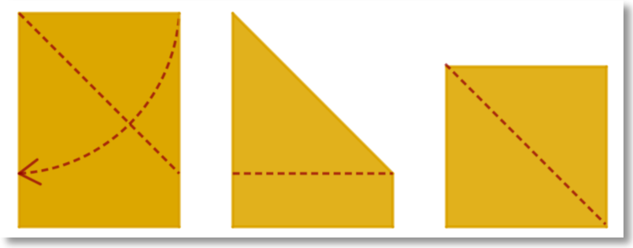


Fig. 3. Illustration of algorithm of paper folding – model of a square

Paper folding enables the verification of some of the properties of a square, such as congruity of sides, symmetry in a square and the perpendicularity of the diagonals. Similarly, the properties of oblongs can be illustrated.

It is possible to investigate the difference between rectangles and other parallelograms using a real model composed of wooden sticks connected with rivets (Figure 4). This model enables changing of the size of the inner angles of the parallelograms. This model can not only distinguish between models and non-models of oblongs but it also creates situations for realising the inclusive relationship between rectangles and parallelograms.

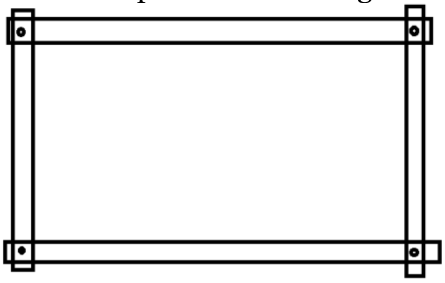


Fig. 4. Illustration of a real model used to distinguish between rectangles and parallelograms

We were able to formulate analogous problems in the dynamic geometry environment. Manipulation of interactive and dynamic applets (Figures 5a–5c), in which the position and metric properties of shape could be changed, enabled us to examine not only the properties of individual shapes but also the inclusive relationships between sets of shapes.

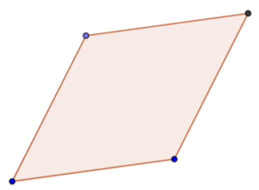


Fig. 5a. Interactive models of rhombuses
Source: https://www.geogebra.org/m/dEgASnxY

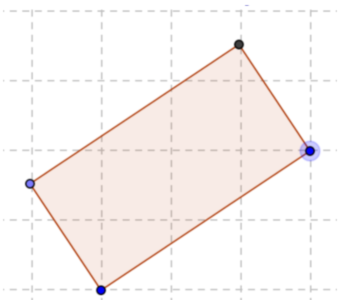


Fig. 5b. Interactive models of rectangles
Source: https://www.geogebra.org/m/ReGb6RvB

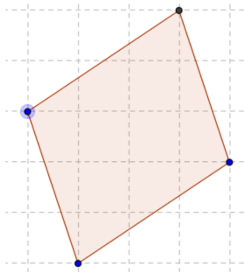


Fig. 5c. Interactive models of parallelograms
Source: https://www.geogebra.org/m/kkk8rgmq

Experimenting with the position of points or with the length of line segments that form the boundary of a quadrilateral allows the pupil the opportunity to examine the properties of the entire class of quadrilaterals, simulate different situations and answer questions or tasks such as: *"Is every oblong a parallelogram? Is every square a parallelogram? Is every rhombus a parallelogram?"* In the mentioned examples, a model from a class of rhombus is created such that changing position and metric properties might lead to the creation of a model of a square as a special type of rhombus.

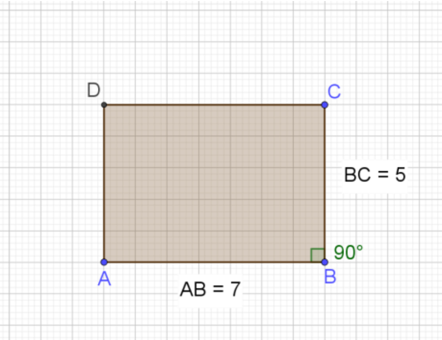


Fig. 6a. Visualisation of properties of rectangles
Source: https://www.geogebra.org/m/gqmqryfy

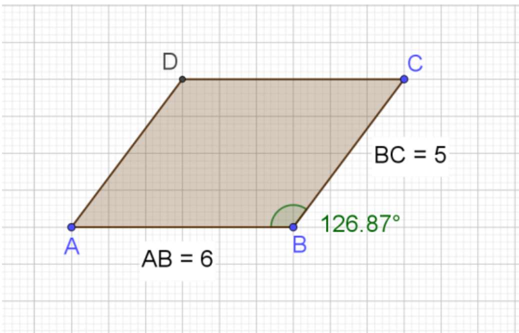


Fig. 6b. Visualisation of properties of parallelograms
Source: https://www.geogebra.org/m/gqmqryfy

Similarly, a model from a class of rectangles can be constructed so that manipulating the vertices of the parallelogram leads to the visualisation of any type of parallelogram (e.g. square, rhombus, oblong, rhomboid). The interaction in the applet concerning rectangles allows the opportunity to model and manipulate oblongs and squares in different positions. In interactive models for older pupils, we recommend also visualising the metric properties (e.g. length of sides, size of angles; Figures 6a and 6b).

An appropriate applet for distinguishing between models and non-models of rectangles is one in which the sizes of all inner angles of polygons are visualised (Figure 7). It is possible to modify this applet for models and non-models of rectangles if we position rectangles and other shapes, where we explain the relationships between their basic elements (e.g. relationships between sides and their length, size of inner angles etc.), into the square grid. These activities may be similarly modelled on a geoboard.

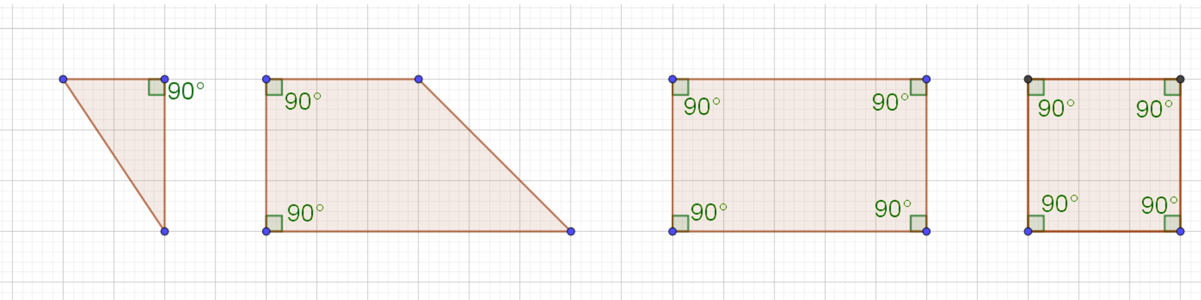


Fig. 7. Visualisation of models and non-models of rectangles
Source: https://www.geogebra.org/m/m7hmkmtc

The results from testing the 4th and 9th graders showed that pupils from the 5th to the 9th grade are insufficiently familiarised with the relationships between basic elements of a square, such as the sides and diagonals. The majority of tested pupils in the 9th grade considered the length of the side and the diagonal of a square as being the same. To eliminate these misconceptions, it is possible to use an applet (Figure 8) that enables the comparison of the lengths of line segments *SQ*

and *SP* (diagonal and side of a square) using motion of the point Q_1. Analogous activity may be undertaken with the use of point P_1 and by comparing the lengths of line segments *RP* and *RS*.
By moving points *S* and *R* it is possible to continuously change the length of the side *RS* and also its position in the plane. Thus, it is possible to show pupils that the length and position of the diagonals of a square *PQRS* are perpendicular, all the inner angles are right angles and the sides are of equal length.

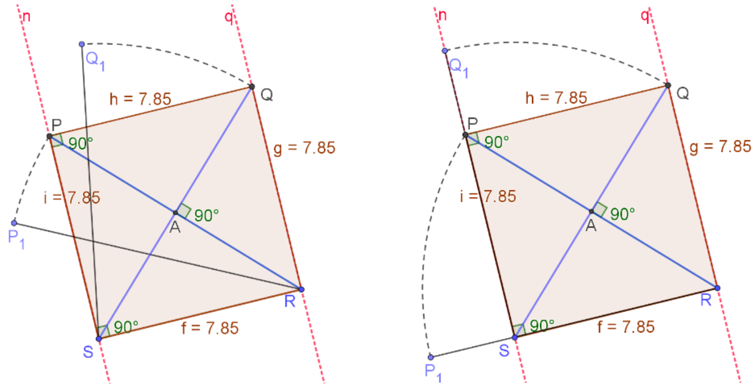


Fig. 8. Applet in GeoGebra software program
Source: https://www.geogebra.org/m/uzfk5xrq

The majority of the tested pupils in the 9th grade had problems with the terms perpendicularity and parallelism of line segments, with only 24 % of these pupils denoting that opposite sides of a square are perpendicular. Thus, we can use an applet in which the work with the square is realised on square ground (Figure 9). Points *R* and *S* can be moved to grid points on the geoboard to create different squares *PQRS*. In all cases, the line segments *PS* and *QR* are parallel. It is appropriate to combine this activity with a concrete manipulative activity on a real geoboard.

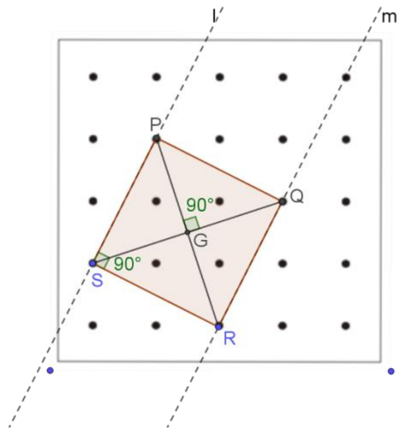


Fig. 9. Square PQRS on geoboard
Source: https://www.geogebra.org/m/mwmv3f4r

3. Conclusion

The results of testing pupils in the 4th and 9th grades (primary level, lower secondary level) showed some misconceptions exist regarding geometric terms and properties despite education. One of the on-going misconceptions is the identification of squares that do not have sides in a vertical and horizontal position. For both groups of pupils, such squares were difficult to identify. It was also shown that a low level of knowledge existed regarding the relationships between basic elements of rectangles (e.g. sides, diagonals) throughout school years, particularly with the comparison of the lengths of sides, diagonals, angles between them, and their perpendicularity and parallelism. Pupils in the 4th and 9th grades also showed decreased abilities to identify rectangles that were not squares. A similar problem was found when distinguishing between squares and rhombuses. Moreover, the 9th graders showed a low ability to distinguish between quadrilaterals that were rectangles and those that were not.

Our research findings led to the projection of activities focused on educational intervention in teaching about rectangles. The aim of the prepared activities and applets presented in the present study was to reduce the misconceptions by pupils about rectangles using dynamic visualisation. These activities focused on visualisation using appropriate educational software programs that should be supported with the use of other activities that utilise real models and appropriate manipulative activities. The more different stimuli and environments a teacher can offer, the earlier the enhancement of their students' geometric conceptions can occur. In ideal cases, pupils would participate in the creation of real and virtual models (for example through project education), which support constructivist approaches in education with an emphasis on enquiry education. As mentioned previously, Duval (1998) discusses the synergy of three processes that are essential for the development of geometric thinking. The suggested manipulative and virtual activities or applets create an educational environment in which visual, constructional and argumentative processes supplement and support each other.

If we want to increase the level of geometric thinking of pupils, it is necessary to personalise education based on the abilities, needs and interests of pupils. Such personalisation may be supported by multiple visual methods shown in the present study. To ensure that this visualisation in education is effective, the choice of the method of visualisation of geometric terms, relationships and construction procedures is subject to the personal educational styles of the pupils and their preferences.

4. Acknowledgements

This study was supported by the Slovak Research and Development Agency project no. APVV-15-0378 (OPTIMAT) "Optimization of mathematics teaching materials based on analysis of the current needs and abilities of pupils of younger school age".

The authors would like to specifically thank Prof. Zalman Usiskin for his permission to use the van Hiele test of geometric thinking in this study.

References

Al-ebous, 2016 – *Al-ebous, T.* (2016). Effect of the van Hiele model in geometric concepts acquisition: The attitudes towards geometry and learning transfer effect of the first three grades students in Jordan. *International Education Studies*, 9(4), 87-98.

Bruner, 1966 – *Bruner, J.S.* (1966). *Toward a theory of instruction*. Cambridge, MA: Harvard University Press.

Burger, Shaughnessy, 1986 – *Burger, W.F., Shaughnessy, M.* (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.

Clements et al., 1999 – *Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., Sarama, J.* (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, 30(2), 192-212.

Clements, Sarama, 2014 – *Clements, D.H., Sarama, J.* (2014). Learning and teaching early Math: The learning trajectories approach. New York, USA: Routledge.

Çontay, Paksu, 2012 – *Çontay, E.G., Paksu, D.A.* (2012). Preservice mathematics teachers' understandings of the class inclusion between kite and square. *Procedia – Social and Behavioral Sciences*, 55, 782-788.

Crowley, 1987 – *Crowley, M.L.* (1987). Learning and teaching geometry. Virginia, USA: NCTM.

Dongwi, 2014 – *Dongwi, B.L.* (2014). Using the van Hiele phases of instruction to design and implement a circle geometry teaching programme in a secondary school in Oshikoto region: A Namibian case study [Electronic resource]. URL: http://repository.unam.na/bitstream/handle/11070/2174/dongwi_van%20hiele_2014.pdf?sequence=1&isAllowed=y

Duval, 1995 – *Duval, R.* (1995). Geometrical pictures: Kinds of representation and specific processes. In R. Sutherland & J. Mason (eds.). Exploiting mental imagery with computers in mathematical education (pp. 142-157). Berlin: Springer.

Duval, 1998 – *Duval, R.* (1998). Geometry from a cognitive point of view. C. Mammana & V. Vilani (Eds.). Perspectives on the Teaching of Geometry for the 21st Century. An ICMI Study. [Electronic resource]. URL: https://books.google.cz/books?id=dOksBAAAQBAJ&printsec=frontcover&hl=cs#v=onepage&q&f=false

Erdogan, Dur, 2014 – *Erdogan, E. O., Dur, Z.* (2014). Preservice mathematics teachers' personal figural concepts and classifications about quadrilaterals. *Australian Journal of Teacher Education*, 39(6), 106-133.

Erikson Institute, 2013 – Erikson Institute. Eriksonmath. Recognizing shapes with Child 10. (2013, February 24). [video file]. [Electronic resource]. URL: http://earlymath.erikson.edu/recognizing-shapes-with-child-10/

Feza, Webb, 2005 – *Feza, N., & Webb, P.* (2005). Assessment standards, Van Hiele levels, and grade seven learners' understanding of geometry. *Pythagoras*, *62*, 36-47.

Fischbein, 1993 – *Fischbein, E.* (1993). The theory of figural concepts. *Educational Studies in Mathematics*, 24(2), 139-162.

Hannibal, 1999 – *Hannibal, M.A.* (1999). *Young children's developing understanding of geometric shapes. The National Council of Teachers of Mathematics.* [Electronic resource]. URL: http://lesage.blogs.uoit.ca/wp-uploads/2010/08/Young-Learners-Geometry_1999-NCTM.pdf

Hejný, 2014 – *Hejný, M.* (2014). Vyučování matematice orientované na budování schémat: aritmetika 1. stupně. Prague, Czech Republic: PF Univerzita Karlova.

Innovated National Educational..., 2015 – *Innovated National Educational Programme for Primary Education (Inovovaný Štátny vzdelávací program Matematika primárne vzdelávanie).* (2015). [Electronic resource]. URL: http://www.statpedu.sk/files/articles/dokumenty/inovovany-statny-vzdelavaci-program/matematika_pv_2014.pdf

Jirotková, 2010 – *Jirotková, D.* (2010). Cesty ke zkvalitňování výuky geometrie. Prague, Czech Republic: Univerzita Karlova v Praze.

Knight, 2006 – *Knight, K.Ch.* (2006). *An Investigation into the change in the van Hiele levels of understanding geometry of pre-service elementary and secondary mathematics teachers.* (Thesis). The University of Maine [Electronic resource]. URL: https://digitalcommons.library.umaine.edu/etd/1361

Levenson et al., 2011 – *Levenson, E., Tirosh, D., Tsamir, P.* (2011). Preschool geometry. Theory, research, and practical perspectives (e-book). Rotterdam, Nederland: Sense Publishers.

Marchis, 2012 – *Marchis, I.* (2012). Preservice primary school teachers' elementary geometry knowledge. *Acta Didactica Napocensia*, 5(2), 33-40.

Mason, 2002 – *Mason, M.* (2002). *The van Hiele levels of geometric understanding. Professional handbook for teachers, geometry: Explorations and applications*. MacDougal Litteil Inc. [Electronic resource]. URL: http://geometryforall.yolasite.com/resources/Mason,%20Marguerite.%20The%20van%20Hiele%20Levels%20of%20Geometric%20Understanding.%202002.pdf

Mayberry, 1983 – *Mayberry, J.* (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, 14(1), 58-69. DOI: 10.2307/748797

McLeod, 2018 – *McLeod, S.* (2018). Jean Piaget's theory of cognitive development [Electronic resource]. URL: https://www.simplypsychology.org/piaget.html

Musser et al., 2001 – *Musser, G. L., Burger, W. F. & Peterson, B. E.* (2001). Mathematics for elementary teachers. (5th edition). New York, USA: John Wiley & Sons.

National Educational Programme, 2009 – National Educational Programme (Štátny vzdelávací program Matematika Príloha ISCED 1, 2009. [Electronic resource]. URL: http://www.statpedu.sk/files/articles/dokumenty/statny-vzdelavaci-program/matematika_isced1.pdf

Piaget, Inhelder, 2010 – *Piaget, J., Inhelder, B.* (2010). Psychologie dítěte. Prague, Czech Republik: Portál.

Using visualization, 2018a – Using visualization in maths teaching (2018). [Electronic resource]. URL: https://www.open.edu/openlearn/education/using-visualisation-maths-teaching/content-section-3

Using visualization, 2018b – Using visualization in maths teaching. Introduction. (2018). [Electronic resource]. URL: https://www.open.edu/openlearn/ocw/mod/oucontent/view.php?id=3170&printable=1

Usiskin, 1982 – *Usiskin, Z.* (1982). Van Hiele levels and achievement in secondary school geometry. CDASSG Project. Chicago, US: The University of Chicago.

Van de Walle, 2001 – *Van de Walle, J.A.* (2001). Geometric thinking and geometric concepts. Elementary and middle school mathematics: Teaching developmentally. 4th ed. Boston: Allyn and Bacon.

Van Hiele, 1986 – *Van Hiele, P.M.* (1986). *Structure and insight: a theory of mathematics education*. Orlando, Fla.: Academic Press.

Van Hiele, 1999 – *Van Hiele, P.M.* (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics*, 5, 310–316.

Žilková et al., 2018 – *Žilková, K., Partová, E., Kopáčová, J., Tkačik, Š, Mokriš, M., Budínová, I., Gunčaga, J.* (2018). Young children`s concepts of geometric shapes. Harlow: Pearson.

Žilková, 2017 – *Žilková, K.* (2017). Reflexia Van Hiele modelu geometrického myslenia. *Dva dni s didaktikou matematiky 2017 (zborník príspevkov)*. Bratislava: FMFI UK.