Modeling and Visualization Process of the Curve of Pen Point by GeoGebra

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Abstract. This study describes the mathematical construction of a real-life model by means of parametric equations, as well as the two- and three-dimensional visualization of the model using the software GeoGebra. The model was initially considered as "determining the parametric equation of the curve formed on a plane by the point of a pen, positioned on an obstacle of height h, during the process of raising the pen vertically to the surface by linearly moving its back-end on the surface." Firstly a solution was sought for this problem in two dimensions. Based on this problem, two additional sub-problems were formed on a plane, and parametric equations were calculated for these sub-problems as well. The curves formed by these parametric equations were then visualized using GeoGebra. In the second stage, the model was improved, and the parametric equation of the curve formed in the space by the pen point as a result of moving the pen's back-end along any function was determined. The curve formed by this parametric equation was also visualized using the GeoGebra 3-D environment. It is expected that determining mathematical concepts and relationships based on real-life models with these types of training tasks, as well as jointly considering the algebraic and geometric representations during the process, will improve the students' perceptions relating to mathematics.
Keywords: modeling; GeoGebra; parametric equation; real-life problems; 3-dimensional modeling

1. Introduction

Mankind has devised and continually developed mathematics due to the necessity of making certain calculations in daily life. Hans Freudenthal suggested that, historically, mathematics found its origins in real-life problems, that aspects of real life were then mathematized, and that formal mathematical information was achieved afterwards. Hans Freudenthal has named this approach the Realistic Mathematics Education (RME) (Altun, 2008). This approach encompasses two main concepts, which are the horizontal mathematization, and the vertical mathematization. Horizontal mathematization involves the mathematical expression of real-life problems in a mathematical sense. In other words, it is the mathematization of real-life models. Vertical mathematization, on the other hand, involves the re-expression of mathematics with the use of symbols (Freudenthal, 1991). In this context, horizontal mathematization makes use of models, graphs and diagrams (Freudenthal, 1991; Streefland, 1991).

A sub-dimension of the RME is modeling (Streefland, 1991). Modeling is the process of creating a model for a problematic situation. In this respect, the “model” refers to a product formed at the end of a process, while “modeling” refers to the process of creating a physical, symbolic, or abstract model for a particular situation (Kertil, 2008). Modeling activities that are performed for problematic situations actually provide mathematics teachers the opportunity for self-development (Lesh & Doerr, 2008).

The mathematical modeling of a real-life problem with Computer Algebra Systems (CAS) and Dynamic Geometry Software (DGS) is considered by researchers focusing on this field as a problem-solving activity that suits the purposes of mathematical learning. In fact, Zbiek and Conner (2006) have indicated that modeling specifically contributes to the understanding of known mathematical concepts, to the learning of new mathematical concepts, to establishing interdisciplinary relationships, and to both the conceptual and procedural development of students through the detailed demonstration of the applicability of mathematical concepts in real-life.

There are various studies demonstrating the importance of dynamic geometry software and computer algebra systems as tools for realistic mathematics education (Aktümen, 2013; Aktümen, Baltaci, & Yildiz, 2011; Aktümen & Kabaca, 2012; Kabaca & Aktümen, 2010; Widjaja & Heck, 2003). In addition to these, numerous studies have been performed on real-life problems in mathematics education (Aydin & Monoghan; Aydin-Unal & Ipek, 2009; Fauzan, Slettenhaar, & Plomp, 2002; Kwon, 2002; Oldknow & Taylor, 2008; Van Den Heuvel-Panhuizen, 2000). Furthermore, there are a gradually increasing number of studies investigating real-life problems with DGS (Aktümen & Kabaca, 2012; Gecü & Özdener, 2010; Gittinger, 2012; Kabaca & Aktümen, 2010; Widjaja & Heck, 2003). At the same time, there are also studies in the literature regarding three-dimensional modeling with DGS (Aktümen, 2013; Aktümen, Baltaci, & Yildiz, 2011; Aktümen, Doruk, & Kabaca, 2012; Oldknow, 2009; Oldknow & Tetlow, 2008).

In recent times, it can be seen that parametric equations are also being used when performing modeling studies with DGS (Aktümen, 2013; Aktümen & Kabaca, 2012; Filler, 2012). We can see many reflections of the concept of parametric equations in daily life. For example, in the manufacture of Computerized Numerical Control (CNC) milling machines, the necessary calculations are performed by using parametric equations (Özel & Inan, 2001). In this study, the GeoGebra version 5.0 Beta has been chosen for the modeling of real-life problems by using parametric equations, as it has the same features as dynamic geometry software and computer algebra systems, and allows for the use of three-dimensional modeling. These modeling processes were developed by solving the following four problems.

1) The three problems for the *x*-*y* axis are specified below.
   a. What is the parametric equation of the curve formed on a plane by the point of a pen, positioned on an obstacle of height *h*, during the process of raising the pen vertically to the surface by linearly moving its back-end on the surface.
   b. By moving the back-end of a pen linearly on a surface, whose front-end is affixed to a ring at a certain height, what is the form and parametric equation of the curve formed by the pen point?
   c. For the angle $(0,\pi)$ that a pen whose front-end (or extension) is affixed to a ring at a certain height forms with the x axis, what is the form and parametric equation of the curve formed by the pen point?
2) The problem for which an answer would be investigated in space was: "By moving on any function the back-end of a pen whose front-end (or extension) is affixed to a ring at a certain height, what is the form and parametric equation of the curve formed by the pen point?

2. Research Methods

In this study*, the modeling process was evaluated separately on both the plane and the space by using a model that had been developed based on real-life. The first three problems were modeled on a plane, while the last problem was modeled to in a space. The modeling and GeoGebra visualization processes for each one of these problems are provided below.

2.1 Modeling Process for an Option of the Problem 1

For the pen which has an obstacle of height *h* in its front, and for which the distance between its back-end to the origin is *a* units, the situation prior to its movement is provided in Figure 1.The back-end of the pen is positioned on point O, while its point (front-end) is positioned on point B.

* Note: A part of this study was presented in the Ninth Mathematics Symposium as a poster presentation on October 20, 2010.

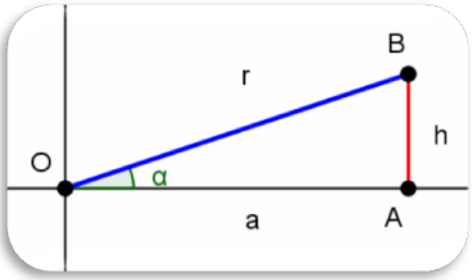


Figure 1. Two-dimensional model of a pen with an obstacle of height *h* in front

According to Figure 1, the first situation is reflected by $r^2 = a^2 + h^2 \rightarrow r = \sqrt{a^2 + h^2}$. The situation resulting from a certain amount of linear movement of the back-end of the pen is provided in Figure 2.

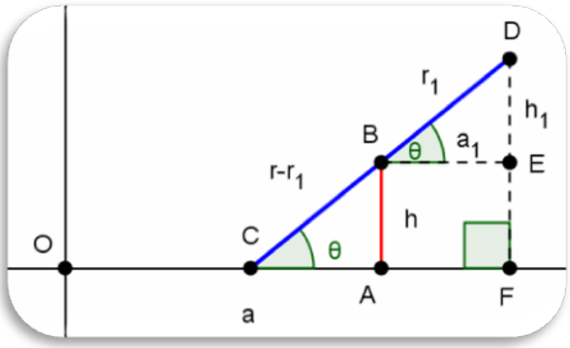


Figure 2. Situation resulting from the linear movement of the back-end of the pen

When the *x* and *y* coordinates of point D are determined accordingly, we obtain:

$$x = a + a_1 \text{ and } y = h + h_1$$

By utilizing triangle BED, we obtain:

$$\cos\theta = \frac{a_1}{r_1},\ \sin\theta = \frac{h_1}{r_1}$$

$$a_1 = r_1 \cos\theta,\ h_1 = r_1 \sin\theta$$

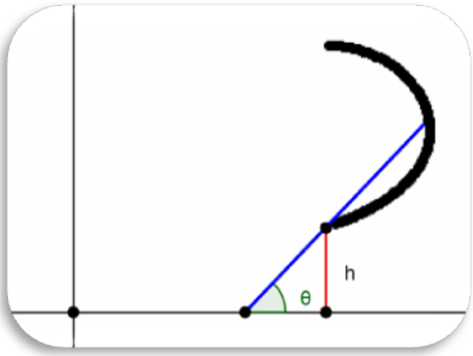
As a result:

$$x = a + r_1 \cos\theta \text{ and } y = h + r_1 \sin\theta \qquad (1)$$

Now, the parametric equation will be obtained when the value of r_1 is calculated. For triangle BCA, $\sin\theta = \frac{h}{r - r_1}$. Thus, $r_1 = r - \frac{h}{\sin\theta}$. In this case, $r_1 = \sqrt{a^2 + h^2} - \frac{h}{\sin\theta}$.

When this value is inserted into the expression provided in (1), the coordinates of D then become $x = a + \left[\sqrt{a^2 + h^2} - \frac{h}{\sin\theta}\right]\cos\theta$ and $y = h + \left[\sqrt{a^2 + h^2} - \frac{h}{\sin\theta}\right]\sin\theta$.

Thus, the parametric equation of the curve formed on a plane by the point of a pen, positioned on an obstacle of height *h*, during the process of raising the pen vertically to the surface by linearly moving its back-end on the surface (Figure 3) is given by:

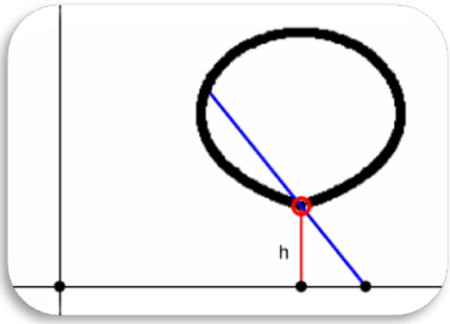


$$x(\theta) = a + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\cos(\theta)$$
$$y(\theta) = h + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\sin(\theta)$$
$$\text{with } \theta \in \left[\arctan(\frac{h}{a}), \frac{\pi}{2}\right]$$

Figure 3. Visualization of the parametric equation for the first problem

2.2 Modeling Process for "b" Option of Problem 1

As a result of the linear movement on the surface of the back-end of a pen affixed to a ring, the angle formed with the *x*-axis assumes values that fall between $\left[\arctan(\frac{h}{a}), \pi - \arctan(\frac{h}{a})\right]$. The parametric equation of the curve formed by the point of the pen (Figure 4) is given by:

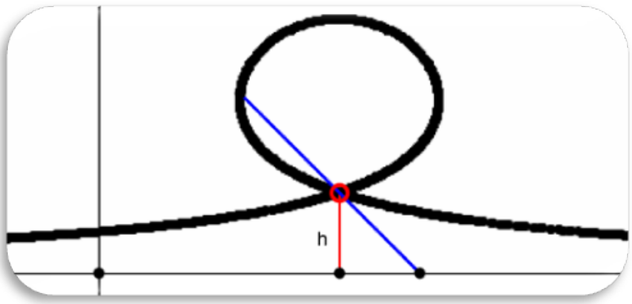


$$x(\theta) = a + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\cos(\theta)$$
$$y(\theta) = h + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\sin(\theta)$$
$$\text{with } \theta \in \left[\arctan(\frac{h}{a}), \pi - \arctan(\frac{h}{a})\right]$$

Figure 4. The modeling and parametric equation for the second situation

2.3 Modeling Process for "c" Option of Problem 1:

By linearly moving on the surface the back-end of the pen affixed to a ring such that its angle with the *x*-axis falls within the $(0, \pi)$ range, the parametric equation of the curve formed by the pen point (Figure 5) becomes:



$$x(\theta) = a + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\cos(\theta)$$
$$y(\theta) = h + \left(\sqrt{a^2 + h^2} - \frac{h}{\sin(\theta)}\right)\sin(\theta)$$
$$\text{with } \theta \in (0, \pi)$$

Figure 5. The modeling and parametric equation for the third situation

2.4 Modeling Process for Problem 2

Figure 6 provides the model for the problem: “By moving on any function the back-end of a pen whose front-end (or extension) is affixed to a ring at a certain height, what is the form and parametric equation of the curve formed by the pen point?”

Figure 6. Three-dimensional description of the model

For resolution of this problem, the labeling described in Figure 7 was employed.

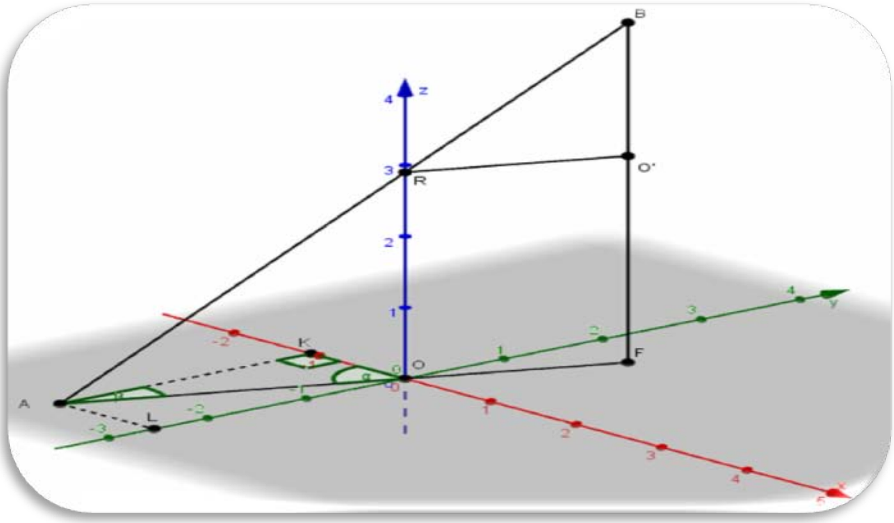


Figure 7. The Formation of the model in three dimensions

With points A, O, F, R, O' and B being planar, $\angle KOA = \alpha$, $\angle KAO = \beta$, $\angle OAR = \theta$, $|OR| = h$, $|AB| = k$, $|OK| = t$, $|OL| = f(t)$ (with $f(x)$ being a function determined by the user), and the line segment AB representing the pen;

$|AO| = \sqrt{t^2 + f(t)^2}$, since $m\angle OKA = 90^\circ$.

Since $m\angle AOR = 90^\circ$, $|AR| = \sqrt{t^2 + f(t)^2 + h^2}$, thus $|RB| = k - \sqrt{t^2 + f(t)^2 + h^2}$.

Since $m\angle OAR = m\angle O'RB$, $|RO'| = |RB|.\cos\theta = \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right).\cos\theta$.

And since ΔARO ≅ ΔRBO' and $\cos\theta = \frac{\sqrt{t^2 + f(t)^2}}{\sqrt{t^2 + f(t)^2 + h^2}}$,

$$|RO'| = \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right).\frac{\sqrt{t^2 + f(t)^2}}{\sqrt{t^2 + f(t)^2 + h^2}}.$$

Let us now determine the coordinates of point B.

Since $|RO'| = |OF|$, $B = (OF.\cos\alpha, OF.\sin\alpha, k.\sin\theta)$.

It is calculated that $\cos\alpha = \frac{-t}{\sqrt{t^2 + f(t)^2}}$, $\sin\alpha = \frac{-f(t)}{\sqrt{t^2 + f(t)^2}}$ and $\sin\theta = \frac{h}{\sqrt{t^2 + f(t)^2 + h^2}}$

Since $|OF| = \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right)\frac{\sqrt{t^2 + f(t)^2}}{\sqrt{t^2 + f(t)^2 + h^2}}$, the *x*, *y* and *z* coordinates of point B are determined as:

$$x(B) = \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right)\frac{\sqrt{t^2 + f(t)^2}}{\sqrt{t^2 + f(t)^2 + h^2}}.\frac{-t}{\sqrt{t^2 + f(t)^2}}$$

$$= \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right)\frac{-t}{\sqrt{t^2 + f(t)^2 + h^2}} = \frac{\left(-kt + t\sqrt{t^2 + f(t)^2 + h^2}\right)}{\sqrt{t^2 + f(t)^2 + h^2}}$$

$$= t\left(1 - \frac{k}{\sqrt{t^2 + f(t)^2 + h^2}}\right)$$

$$y(B) = \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right)\frac{\sqrt{t^2 + f(t)^2}}{\sqrt{t^2 + f(t)^2 + h^2}}.\frac{-f(t)}{\sqrt{t^2 + f(t)^2}}$$

$$= \left(k - \sqrt{t^2 + f(t)^2 + h^2}\right)\frac{-f(t)}{\sqrt{t^2 + f(t)^2 + h^2}} = \frac{\left(-kf(t) + f(t)\sqrt{t^2 + f(t)^2 + h^2}\right)}{\sqrt{t^2 + f(t)^2 + h^2}}$$

$$= f(t)\left(1 - \frac{k}{\sqrt{t^2 + f(t)^2 + h^2}}\right)$$

$$z(B) = k\frac{h}{\sqrt{t^2 + f(t)^2 + h^2}}$$

Provided below are the curves formed by the point of the pen for certain functions in which its back-end is moved.

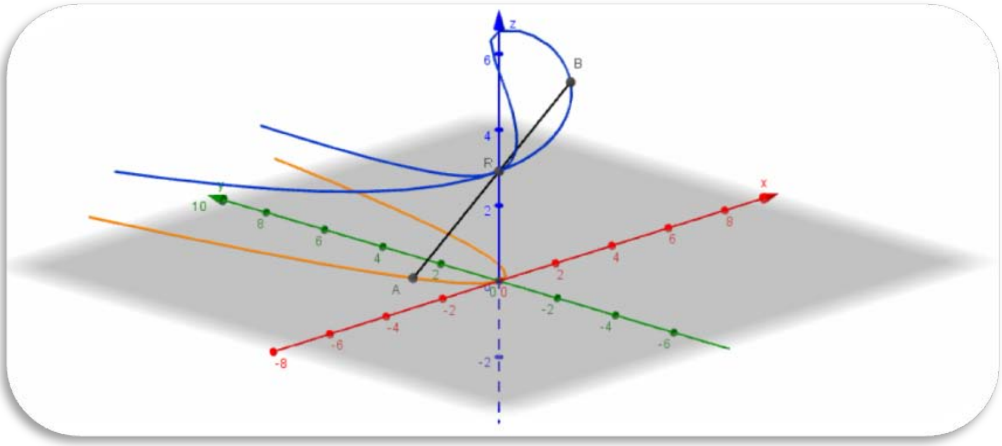


Figure 8. Curve formed as a result of the movement of the back-end of the pen on the $f(x) = x^2$ function

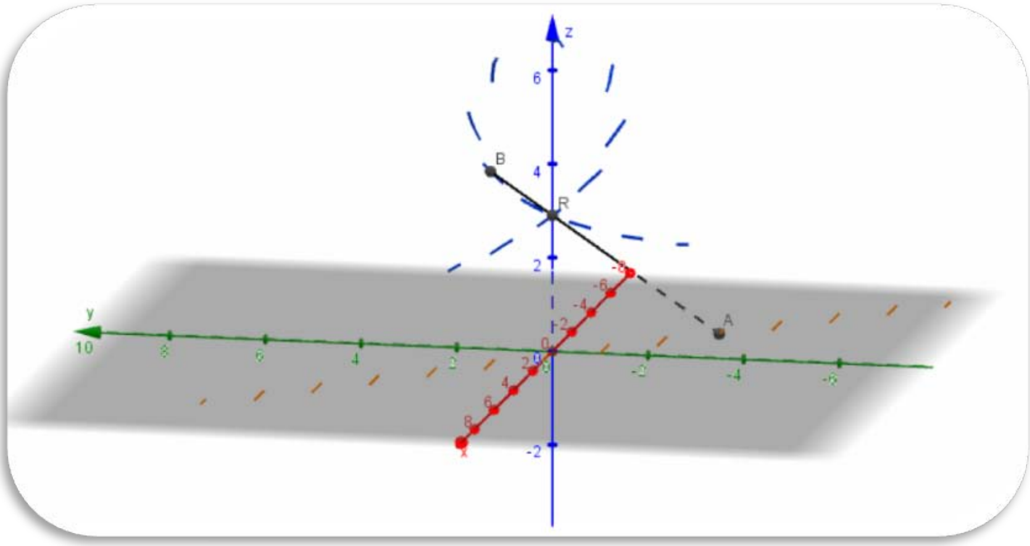


Figure 9. Curve formed as a result of the movement of the back-end of the pen on the $f(x) = \|x\|$ function

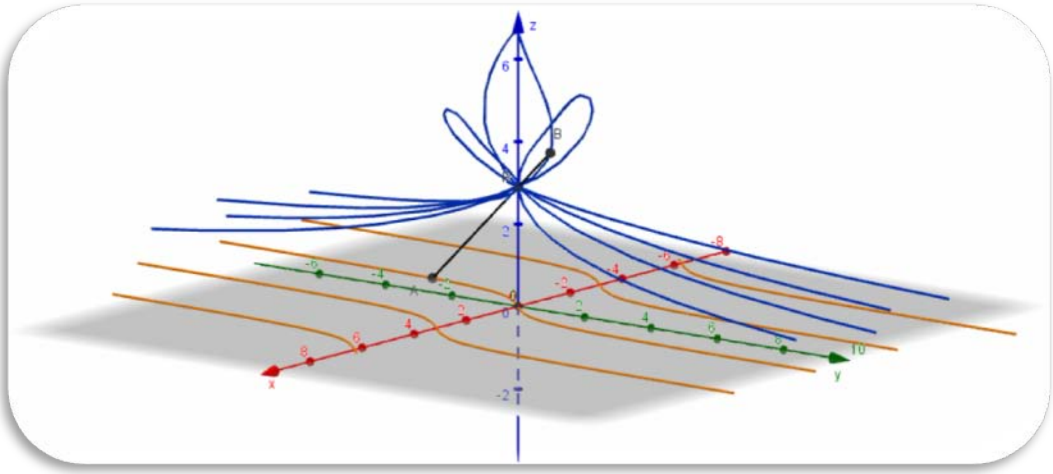


Figure 10. Curve formed as a result of the movement of the back-end of the pen on the $f(x) = \tan x$ function

3. Conclusions

In this study, the modeling processes for real-life problems were described by forming problem situations based on a real-life model. At the end of these processes, the parametric equations of the curve created by the pen point curve were formulated both for the plane and the space. Visualization of these processes was ensured by using the GeoGebra 5.0 Beta software, which is dynamic geometry software. It is expected that determining mathematical concepts and relationships based on real-life models with these types of training tasks, and jointly considering the algebraic and geometric representations during the process, will improve the overall students' perceptions of mathematics. In fact, Freudenthal has expressed that it is necessary to associate learning in mathematics classes with real-life, and that sustaining this approach would be one of the most suitable methods to follow (Gravemeijer & Terwel, 2000; Muijs & Reynolds, 2011; Wubbels, Korthagen, & Broekman, 1997). It can thus be argued that a dynamic model pertaining to a real-life problem can assist us in explaining and interpreting mathematical models, and thereby support a better understanding of a mathematical model by demonstrating its graphical representation and relationships (Doerr & Pratt, 2008; Duval, 1999). We are suggesting that the GeoGebra 5.0 Beta software can provide a suitable environment for designing such models, and that by using this software students become more engaged in their mathematics learning. We also contend that developing such models can assist students who have difficulties thinking in 3-dimensions in terms of developing their spatial skills.

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